ALGEBRAIC GEOMETRY

SMI PERUGIA

2026

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Syllabus: This course is an introduction to algebraic geometry, from the point of view of schemes, with a concrete view based on examples, closely related to affine and projective varieties over a field.

We will first look at affine schemes. These are defined in general starting from commutative rings by considering the set of all prime ideals equipped with the Zariski topology. The turning point is the introduction of the sheaf of regular functions on this topological space. Examples here are closely related to affine algebraic varieties.

Then we will turn to more general schemes, as spaces equipped with a sheaf of rings that locally looks like the regular functions on an affine scheme. We will see how schemes are equipped with a package of data such as local rings of germs of regular functions, residue fields, how subschemes and morphisms of schemes look like.

Next, we will analyse some basic properties of schemes, such as reduced structures, irreducible components and their generic points, their dimension and finiteness conditions. We will look at separated / proper schemes, which are algebraic versions of the ideas of Hausdorff / compact space. We will also study schemes in families, fibre products, base change and various applications to algebraic varieties.

Text book: The course is based on [EH00]. Most of the notions are developed in the first four chapters of this book.

We will also use some portions of [Liu02], notably the material contained in the first four chapters. Most of the exercices and proofs can be found in one of these two textbooks, or in both.

Prerequisites: Here are some prelimilary notions useful for this course. They are divided into three sets, namely mandatory / desirable / more advanced skills.

I will assume that the students are familiar with the following notions.

Groups, rings, fields, algebras. Homomorphisms of these algebraic structures. Field extensions, algebraic and transcendental elements. Characteristic of a field. The fraction field of an integral domain. Ideals of a commutative ring, maximal ideals, prime ideals, quotient rings, principal ideal domains, factorial rings, Noetherian rings. Geometry of the affine space over a field.

Further desirable background knowledge.

Modules over a ring (finitely generated / finitely presented modules, annihilators, projective modules). Localization of rings and modules (localization at an element / at a prime ideal, relationship with the fraction field). Nilpotent elements, zero divisors of a ring. Basic notions of projective geometry (homogeneous coordinates, linear subspaces, affine charts).

More advanced topics that may be of help, or suggested preliminary reading.

The algebraic set associated with an ideal of a polynomial ring (the radical of an ideal, Nullstellensatz). Tensor product of vector spaces, of modules over a ring, of algebras over a ring (construction and universal property). Sheaves on a topological space, definitions and examples (the cocycle condition; sheaves of continuous functions, of differential forms on a smooth manifold, etc). Some categorical language (functors, adjoint pairs).

References

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications.