Geometric Flows

The heat flow for harmonic maps of surfaces is a prototypical example of a geometric evolution problem. We review the classical results on existence, uniqueness, and finite-time blow-up as well as on the characterization of singularities, and we show how a monotonicity formula and ϵ-regularity result for the flow allow to obtain partially regular weak solutions in the higher-dimensional case.

Back in the two-dimensional setting we then show how the monotonicity formula may be used to obtain topological information at blow-up points when combined with a pointwise form of the energy identity and a Łojasiewicz inequality, thus also making contact with the second main theme of the school.

As a further example of a geometric flow we finally consider the recently defined Plateau flow, exhibiting a surprisingly close relationship with the harmonic map heat flow.

Along the way we indicate numerous open problems that invite further study.

1. Harmonic maps
   - Setting, definitions, Bochner identity
   - Some existence and non-existence results, $m = 2$
2. The heat flow
   - Local existence
   - Energy inequality, Bochner identity
3. Global existence and uniqueness, $m = 2$
   - Statement and proof of the main theorem
   - Remarks on uniqueness
   - Equivariant maps $B \to S^2$; global existence and finite-time blow-up
   - Remarks on higher genus
4. Higher dimensions
   - Monotonicity formula, ϵ-regularity
   - Global existence of partially regular weak solutions
   - Normalized harmonic map heat flow on spheres
5. Fine analysis of blow-up for the heat flow of maps $B \to S^2$
   - Pointwise energy inequality, monotonicity formulas
   - Continuity of “body map”
6. Plateau flow
   - Plateau’s problem
   - Global existence and uniqueness of Plateau flow
   - Remarks on higher genus

Quantitative estimates for geometric variational problems

To understand the key features of a variational problem it is crucial to not only analyse exact minimisers and critical points of a given quantity E, but also maps that almost minimise E or that almost solve the associated Euler-Lagrange equation $\nabla E(u) = 0$. In particular, one needs to understand whether, and in what sense, such almost solutions provide good approximations to exact solutions. A fundamental goal in the quantitative study of variational problems and PDEs is hence to obtain

- *Łojasiewicz estimates* that control the distance to critical points/values in terms of $\|\nabla E(u)\|$;
- *quantitative stability estimates*, that bound the distance of almost minimisers to the nearest minimiser in terms of the defect $\delta_u = E(u) - E_{\min}$.

Łojasiewicz estimates are invaluable tools also in the analysis of gradient flows and are closely connected to deep questions regarding the existence of solutions and the structure of the energy spectrum. Starting from the foundational paper of Leon Simon [8] we will first present the classical functional analytic approach to Łojasiewicz estimates, then discuss its limitations and challenges and finally cover recent developments that allow to obtain and apply Łojasiewicz estimate in the presence of singularities and multi-scale behaviour.

Over the past two decades, the quantitative analysis of almost minimisers has become a core topic in the analysis of PDEs and variational problems and we will cover some aspects of this theory, starting with the classical paper of Bianchi and Egnell [1] on optimal Sobolev embeddings, discussing how both elliptic regularisation techniques and gradient flows can be used to establish sharp bounds and how quantitative, and even qualitatitve, stability of minimisers can break down in the presence of singularities or multiscale behaviour.

1. Introduction
2. Łojasiewicz estimates in regular settings
    - Basic functional analytic tools
    - Classical approach of Simon
    - Limitations and Extensions
    - Applications to asymptotic analysis of gradient flows
3. Quantitative stability of minimisers in regular settings
    - Classical approach and results
    - Elliptic regularisation and the selection principle
    - Gradient flow methods
4. Łojasiewicz estimates in singular settings
    - Break down of classical methods
    - Use of dimension reduction techniques to reduce the problem to construction of models
    - Almost harmonic maps from surfaces into general manifolds
5. Quantitative stability estimates for maps into S^2
    - Classical results for degree 1 maps
    - Breakdown of classical stability results in presence of multiscale behaviour
    - Generalised quantitative stability of almost minimisers exhibiting multiscale behaviour

Possible topics for seminars:

- Proof of the Sacks-Uhlenbeck [20] existence result for harmonic maps from a closed surface to a target N with $\pi_2(N) = 0$, based on [22] or [11].
- Global existence for equivariant maps $B \to S^2$ of degree $k \geq 2$, based on [18] and [21].
- Non-uniqueness for weak solutions of harmonic map flow violating energy monotonicity, based on [23].
- Łojasiewicz estimates for maps from S^2 to S^2 (based on a selection of material from the papers [25, 24] of Topping and the recent paper [26] of Waldron)
- Applications of dimension reduction techniques for the H-surface equation on bounded domains (based on paper [14] of Chanillo-Malchiodi)
- Quantitative stability for minimizing Yamabe metrics (based on paper [17] of Engelstein, Neumayer and Spolaor)
- More general functional analytic framework for Łojasiewicz estimates (based on papers of Chill [15] and Rupp [19])
- Quantitative stability results for conformal immersions (based on the paper [16] of De Lellis and Müller).

Prerequesites:

Basic knowledge of Functional Analysis, Sobolev spaces, (linear) theory of elliptic and parabolic PDEs and Differential Geometry, as covered in a Master's degree in mathematics.

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