Functional Analysis

Jarosław Mederski

April 28, 2025

Scuola Matematica Interuniversitaria, Perugia 2025

This course serves as an introduction to Functional Analysis with applications to elliptic partial differential equations (PDEs). It focuses on the fundamental theorems concerning Banach and Hilbert spaces, along with selected applications. The theoretical material will be supported by numerous examples drawn from mathematical physics. Moreover, recent directions of scientific research in the area of semilinear partial differential equations, as well as emerging challenges where functional analysis plays a crucial role, will also be discussed. The course is designed to be largely self-contained, and complete lecture notes will be provided to students.

Contents

1 Banach and Hilbert Spaces

- Normed spaces and complete norms, examples
- Bounded linear operators
- Hahn-Banach theorem
- Strong and weak convergence
- Weak* topology and Banach-Alaoglu theorem
- Duals of normed spaces
- Reflexive spaces
- Open Mapping theorem
- Closed Graph theorem
- Adjoint operators, compact operators and their spectrum
- Hilbert spaces, projections and orthogonality
- Riesz representation theorem
- Spectral decomposition of self-adjoint compact operators
- Convexity and coercivity

2 Sobolev Spaces and elliptic PDEs

- Sobolev spaces
- Embeddings
- The spectrum of the Laplace operator on a bounded domain
- Variational characterization of the first eigenvalue
- Weak solutions and critical points
- Linear and nonlinear Dirichlet problems
- Coercive problems and minimization techniques
- Minimax methods and the Mountain Pass Theorem
- Solutions to semilinear elliptic PDEs

References

[1] Marino Badiale, Enrico Serra: *Semilinear Elliptic Equations for Beginners: Existence Results via the Variational Approach*, Springer 2011.

[2] Haïm Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer 2011.

[3] Walter Rudin: *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.